HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

1 Higher–Order Differential Equations

Consider the differential equation: $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)\, y' + p_0(x)\, y = 0$

General Solution

A general solution of the above n^{th} order *homogeneous* linear differential equation on some interval I is a function of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

where $y_1, \ldots, y_n$ are linearly independent solutions (basis) on I.

Theorem – Existence and Uniqueness of IVP

If the $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ in the differential equations are continuous on an open interval I, then the initial value problem [with x_0 in I] has a unique solution in I.

Wronskian

The *Wronskian* of $y_1, y_2, \ldots, y_n$ is defined as

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Theorem – Linear Dependence and Independence of Solutions

Let $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ be continuous in I, $[x_0, x_1]$, and let $y_1, y_2, \ldots, y_n$ be n solutions of the differential equation. Then

(1) $W(y_1, y_2, \ldots, y_n)$ is either zero for all $x \in I$ or for no value of $x \in I$.

(2) $y_1, y_2, \ldots, y_n$ are linearly independent if and only if

$$W(y_1, y_2, \ldots, y_n) \neq 0$$

Theorem – Existence of a General Solution

Theorem – General Solution

[Exercise] Consider the third–order equation

$$y''' + a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$$

where a, b and c are continuous functions of x in some interval I. The Wronskian of $y_1(x)$, $y_2(x)$, and $y_3(x)$ is defined as

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

where y_1, y_2 and y_3 are solutions of the differential equation.

(a) Show that W satisfies the differential equation $W' + a(x)\,W = 0$

(b) Prove that W is always zero or never zero.

(c) Can you extend the above results to n^{th}–order linear differential equations?

2 nth-Order Homogeneous Equations with Constant Coefficients

$$y^{(n)} + a_{n-1}\, y^{(n-1)} + \ldots + a_1\, y' + a_0\, y = 0$$ Differential Equation

$$\lambda^n + a_{n-1}\, \lambda^{n-1} + \ldots + a_1\, \lambda + a_0 = 0$$ Characteristic Equation

Case I ***Distinct Roots, $\lambda_1, \lambda_2, \ldots, \lambda_n$***

The corresponding linearly independent solutions are

$$e^{\lambda_1 x},\ e^{\lambda_2 x},\ \ldots,\ e^{\lambda_n x}$$

Case II ***Multiple Roots, $\lambda_1 = \lambda_2 = \ldots = \lambda_m = \lambda$***

The corresponding linearly independent solutions are

$$e^{\lambda x},\ x\, e^{\lambda x},\ x^2\, e^{\lambda x},\ \ldots,\ x^{m-1}\, e^{\lambda x}$$

Case III ***Complex Simple Roots $\lambda_1 = \gamma + i\,\omega$, $\lambda_2 = \gamma - i\,\omega$***

The corresponding linearly independent solutions are

$$e^{\gamma x} \cos \omega x \quad , \quad e^{\gamma x} \sin \omega x$$

Case IV　　*Complex Multiple Roots*

$$\lambda_1 = \lambda_3 = \lambda_5 = \ldots = \lambda_{2m-1} = \gamma + i\,\omega$$

$$\lambda_2 = \lambda_4 = \lambda_6 = \ldots = \lambda_{2m} = \gamma - i\,\omega$$

The corresponding linearly independent solutions are

$$e^{\gamma x} \cos \omega x, \; x\, e^{\gamma x} \cos \omega x, \; \ldots, \; x^{m-1} e^{\gamma x} \cos \omega x$$

$$e^{\gamma x} \sin \omega x, \; x\, e^{\gamma x} \sin \omega x, \; \ldots, \; x^{m-1} e^{\gamma x} \sin \omega x$$

[Example] $y''' - 3y'' - 10y' + 24y = 0$

[Solution] The characteristic equation is $\lambda^3 - 3\lambda^2 - 10\lambda + 24 = 0$ or $(\lambda - 2)(\lambda + 3)(\lambda - 4) = 0$

$$\lambda = 2,\ -3,\ 4 \quad \text{(Case I)}$$

$$\Rightarrow \quad y = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{4x}$$

[Example] $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$

[Solution] The characteristic equation is $\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = 0$ or $(\lambda - 1)^4 = 0$

$$\lambda = 1,\ 1,\ 1,\ 1 \quad \text{(Case II)}$$

$$\Rightarrow \quad y = c_1 e^{x} + c_2 x e^{x} + c_3 x^2 e^{x} + c_4 x^3 e^{x}$$

[Example] $y^{(5)} - 2y^{(4)} + 8y'' - 12y' + 8y = 0$

[Solution] The characteristic equation is $\lambda^5 - 2\lambda^4 + 8\lambda^2 - 12\lambda + 8 = 0$ or $(\lambda + 2)(\lambda^2 - 2\lambda + 2)^2 = 0$

$$\lambda = -2,\ 1+i,\ 1-i,\ 1+i,\ 1-i \quad \text{(Case IV)}$$

$$\Rightarrow \quad y = c_1 e^{-2x} + c_2 e^{x} \cos x + c_3 x e^{x} \cos x + c_4 e^{x} \sin x + c_5 x e^{x} \sin x$$

[Exercise 1] Reduction of Order of Higher–Order Equations

[Exercise 2] Consider the third–order equation

$$y''' + a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$$

and let $y_1(x)$ and $y_2(x)$ be two given linearly independent solutions.

Define $y_3(x) = v(x)\, y_1(x)$ and assume that y_3 is a solution to the equation.

(a) Find a second–order differential equation that is satisfied by v'.
(b) Show that $(y_2/y_1)'$ is a solution of this equation.
(c) Use the result of part (b) to find a second, linearly independent solution of the equation derived in part (a).

[Exercise 3] [Euler-Cauchy Equation of the Third Order] The Euler equation of the third order is

$$x^3 y''' + a\, x^2 y'' + b\, x\, y' + c\, y = 0$$

Show that $y = x^m$ is a solution of the equation if and only if m is a root of the characteristic equation

$$m^3 + (a - 3)\, m^2 + (b - a + 2)\, m + c = 0$$

What is the characteristic equation for the n^{th} order Euler equation?

2 Nonhomogeneous Equations

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)\, y' + p_0(x)\, y \;=\; r(x)$$

$$\Rightarrow \qquad y(x) \;=\; y_h(x) + y_p(x)$$

where again $y_h(x) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ is a general solution of the homogeneous equation and $y_p(x)$ is a particular solution to the nonhomogeneous equation.

(1) Method of Undetermined Coefficients

Same as in the Chapter 2.

In summary, for a constant coefficient nonhomogeneous linear differential equation of the form

$$y^{(n)} + a\,y^{(n-1)} + \ldots + f\,y' + g\,y = r(x)$$

we have the following rules for the method of undetermined coefficients:

(A) ***Basic Rule:*** *If $r(x)$ in the nonhomogeneous differential equation is one of the functions in the first column in the following table, choose the corresponding function y_p in the second column and determine its undetermined coefficients by substituting y_p and its derivatives into the nonhomogeneous equation.*

(B) ***Modification Rule:*** *If any term of the suggested solution $y_p(x)$ is the solution of the corresponding homogeneous equation, multiply y_p by x repeatedly until no term of the product $x^k y_p$ is a solution of the homogeneous equation. Then use the product $x^k y_p$ to solve the nonhomogeneous equation.*

(C) ***Sum Rule:*** *If $r(x)$ is sum of functions listed in several lines of the first column of the following table, then choose for y_p the sum of the functions in the corresponding lines of the second column.*

Table for Choosing y_p

$r(x)$	$y_p(x)$
$P_n(x)$	$a_0 + a_1 x + \ldots + a_n x^n$
$P_n(x)\, e^{ax}$	$(a_0 + a_1 x + \ldots + a_n x^n)\, e^{ax}$
$P_n(x)\, e^{ax} \sin bx$ or/and $Q_n(x)\, e^{ax} \cos bx$	$(a_0 + a_1 x + \ldots + a_n x^n)\, e^{ax} \sin bx$ + $(c_0 + c_1 x + \ldots + c_n x^n)\, e^{ax} \cos bx$

where $P_n(x)$ and $Q_n(x)$ are polynomials in x of degree n ($n\ \varepsilon\ 0$).

EXAMPLE 1 **Initial value problem. Modification Rule**

Solve the initial value problem

$$y''' + 3y'' + 3y' + y = 30e^{-x}, \qquad y(0) = 3, \qquad y'(0) = -3, \qquad y''(0) = -47. \tag{6}$$

Solution. 1st Step. The characteristic equation is $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$. It has the triple root $\lambda = -1$. Hence a general solution of the homogeneous equation is

$$\begin{aligned} y_h &= c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x} \\ &= (c_1 + c_2x + c_3x^2)e^{-x}. \end{aligned}$$

2nd Step. If we try $y_p = Ce^{-x}$, we get $-C + 3C - 3C + C = 30$, which has no solution. Try Cxe^{-x} and Cx^2e^{-x}. The Modification Rule calls for

$$y_p = Cx^3e^{-x}.$$

Then

$$\begin{aligned} y_p' &= C(3x^2 - x^3)e^{-x}, \\ y_p'' &= C(6x - 6x^2 + x^3)e^{-x}, \\ y_p''' &= C(6 - 18x + 9x^2 - x^3)e^{-x}. \end{aligned}$$

Substitution of these expressions into (6) and omission of the common factor e^{-x} gives

$$C(6 - 18x + 9x^2 - x^3) + 3C(6x - 6x^2 + x^3) + 3C(3x^2 - x^3) + Cx^3 = 30.$$

The linear, quadratic, and cubic terms drop out, and $6C = 30$. Hence $C = 5$. This gives $\boxed{y_p = 5x^3e^{-x}.}$

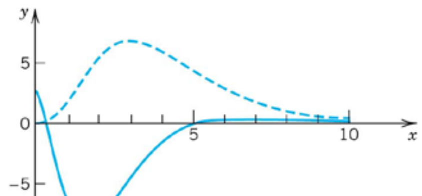
3rd Step. We now write down $y = y_h + y_p$, the general solution of the given equation. From it we find c_1 by the first initial condition. We insert the value, differentiate, and determine c_2 from the second initial condition, insert the value, and finally determine c_3 from $y''(0)$ and the third initial condition:

$$\begin{aligned} y &= y_h + y_p = (c_1 + c_2x + c_3x^2)e^{-x} + 5x^3e^{-x}, && y(0) = c_1 = 3 \\ y' &= [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x}, && y'(0) = -3 + c_2 = -3, \quad c_2 = 0 \\ y'' &= [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x}, && y''(0) = 3 + 2c_3 = -47, \quad c_3 = -25. \end{aligned}$$

Hence the answer of our problem is (Fig. 71)

$$y = (3 - 25x^2)e^{-x} + 5x^3e^{-x}.$$

The dashed curve in Fig. 71 is y_p. Does the curve of y agree with what you can expect from the initial conditions? From the limit as $x \to \infty$?



(2) Method of Variation of Parameters

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)\, y' + p_0(x)\, y = r(x)$$

Given $\Rightarrow$ $y_h = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$

Assume $\mathbf{y_p = u_1 y_1 + u_2 y_2 + \ldots + u_n y_n}$

where u_1, ..., u_n are functions of x. Since the particular solution satisfies the non-homogeneous differential equation, we have

$$y_p^{(n)} + p_{n-1}(x)\, y_p^{(n-1)} + \ldots + p_1(x)\, y_p' + p_0(x)\, y_p = r(x)$$

Now $y_p' = u_1' y_1 + \ldots + u_n' y_n + u_1 y_1' + \ldots + u_n y_n'$

Assume $\mathbf{u_1' y_1 + u_2' y_2 + \ldots + u_n' y_n = 0}$ then $y_p' = u_1 y_1' + \ldots + u_n y_n'$

and $y_p'' = u_1' y_1' + \ldots + u_n' y_n' + u_1 y_1'' + \ldots + u_n y_n''$

Again, we **assume**

$\mathbf{u_1' y_1' + u_2' y_2' + \ldots + u_n' y_n' = 0}$ we have $y_p'' = u_1 y_1'' + \ldots + u_n y_n''$

After differentiation n times, we have a set of simultaneous differential equations of u_1', u_2' . . ., u_n':

$$\begin{aligned}
&y_1u_1' + y_2u_2' + y_3u_3' + \ldots + y_nu_n' = 0\\
&y_1'u_1' + y_2'u_2' + y_3'u_3' + \ldots + y_n'u_n' = 0\\
&y_1''u_1' + y_2''u_2' + y_3''u_3' + \ldots + y_n''u_n' = 0\\
&\ldots\\
&y_1^{(n-2)}u_1' + y_2^{(n-2)}u_2' + y_3^{(n-2)}u_3' + \ldots + y_n^{(n-2)}u_n' = 0\\
&y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + y_3^{(n)}u_3' + \ldots + y_n^{(n-1)}u_n' = r(x)
\end{aligned}$$

The solutions of u_1', u_2', ..., u_n' are

$$u_1' = \frac{1}{W}\begin{vmatrix} 0 & y_2 & \dots & y_n \\ 0 & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ r(x) & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} = \frac{W_1}{W}\ r(x)$$

$$u_2' = \frac{1}{W}\begin{vmatrix} y_1 & 0 & \dots & y_n \\ y_1' & 0 & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & r(x) & \dots & y_n^{(n-1)} \end{vmatrix} = \frac{W_2}{W}\ r(x)$$

.....

$$u_n' = \frac{1}{W}\begin{vmatrix} y1 & y_2 & \dots & 0 \\ y_1' & y_2' & \dots & 0 \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & r(x) \end{vmatrix} = \frac{W_n}{W}\ r(x)$$

Thus, we have

$$y_p(x) = y_1 \int \frac{W_1}{W}\ r(x)\,dx + y_2 \int \frac{W_2}{W}\ r(x)\,dx + \ . \ . \ . \ + y_n \int \frac{W_n}{W}\ r(x)\,dx$$

[Example] $x^3y''' - 4x^2y'' + 8xy' - 8y = 6x^3(x^2+1)^{-3/2}, \qquad x > 0$

[Solution] **Important!** We must re-write the above equation in the standard form:

$$y''' - \frac{4}{x}\,y'' + \frac{8}{x^2}\,y' - \frac{8}{x^3}\,y = r(x) = 6(x^2+1)^{-3/2}$$

The solution of the corresponding homogeneous equation is

$$y_h = c_1 x + c_2 x^2 + c_3 x^4$$

or $\quad y_1 = x, \quad y_2 = x^2, \quad y_3 = x^4$

$$\Rightarrow \quad W(x) = \begin{vmatrix} x & x^2 & x^4 \\ 1 & 2x & 4x^3 \\ 0 & 2 & 12x^2 \end{vmatrix} = 6x^4 \ (\neq 0 \text{ on } (0, \infty))$$

$$u_1' = \frac{1}{W}\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ r(x) & y_2'' & y_3'' \end{vmatrix} = \frac{1}{W}\begin{vmatrix} 0 & x^2 & x^4 \\ 0 & 2x & 4x^3 \\ 6(x^2+1)^{-3/2} & 2 & 12x^2 \end{vmatrix} = \frac{12x^5\left(x^2+1\right)^{-3/2}}{6x^4} = 2x(x^2+1)^{-3/2}$$

$$u_2' = -3(x^2+1)^{-3/2}$$

$$u_3' = x^{-2}(x^2+1)^{-3/2}$$

After integration, we have

$$u_1 = \frac{-2}{(x^2+1)^{1/2}}$$

$$u_2 = \frac{-3x}{(x^2+1)^{1/2}}$$

$$u_3 = \frac{2x^2+1}{x\,(x^2+1)^{1/2}}$$

$$\therefore \quad y_p = u_1 y_1 + u_2 y_2 + u_3 y_3 = -2x\,(x^2+1)^{-3/2}$$

$$\Rightarrow \quad y = c_1 x + c_2 x^2 + c_3 x^4 - 2x\,(x^2+1)^{-3/2}$$